IV Contradictoriness of the empty and the transfinite set[1]

April 2022. The empirical proof is provided that the axiom that requires the empty set contradicts reality. The contradiction with nature also applies to Cantor's idea of the transfinite, as Hilbert has acknowledged. Accordingly, empty and transfinite sets are based on rationalistic theories that contradict empiricism and are therefore necessarily wrong. However, Hilbert did not draw this conclusion but demanded the mathematical consistency of the transfinite.
The formal proof that the transfinite set does not exist requires the refutation of the ZFC-axiom of infinity. This axiom presupposes the empty set, that according to empirical evidence does not exist. The axiom of infinity is therefore contradictory.
The empty set does not appear in Cantor's doctrine, ZFC contradicts Cantor.

1 Introduction

The axiomatic system ZFC of Ernst Zermelo and Abraham Fraenkel requires the existence of the empty set and of transfinite sets.
The empty set $\varnothing$ is defined as a set with no elements, but is itself an element. $\varnothing$ can be united with the elements of any set and then increases the set.
(1) $\{ \varnothing, x_1, x_2, ..., x_n \} > \{ x_1, x_2, ..., x_n \}$
The existence of the transfinite set presupposes the empty set as the 1st element.
Fraenkel stated that set theory is valid "not only for abstract elements", but also for "apples, pears and apricots". However, he has not verified whether the empty and the transfinite set satisfy reality. The review will be conducted here.

2. Current theory of the empty set

Zermelo axiomatically introduced the empty set as an "improper" set. This characterization alone raises doubts. The axiom is formally defined by (2):
(2) Axiom: $\exists \varnothing: \forall x: \neg (x \in \varnothing)$
The empty set $\varnothing$ exists: For all elements x applies: they are not elements of $\varnothing$.
Fraenkel's justification of the empty set is instructive. He argues: "For purely formal reasons we introduce ..... an improper set, …. the empty crowd. According to Cantor's definition, that's not a set..... **The definition is a statement to be judged purely on moments of expediency.** …. None of these stipulations have anything to do with 'correctness' if they are admissible without contradiction." **Fraenkel formally requires the empty set to ensure the transfinite set.**
The empty set is reviewed empirically using apples as an example.

3. Empirical refutation of the empty set

A set without apples that increases a set of apples is a contradiction in itself, the set remains the same.
(3) $\{ \varnothing, \text{apple}_1, \text{apples}_2, ..., \text{apple}_n \} = \{ \text{apple}_1, \text{apple}_2, ..., \text{apple}_n \}$
ZFC's requirement (1) that $\varnothing$ increases the set is thus refuted. The only consistent meaning is "no apple", i.e. "nothing" of the set of apples. This argumentation applies to arbitrary elements.

The empty set does not exist. The axiom (2) is empirically refuted.
The existence of the empty set has also been denied by many mathematicians, logicians and philosophers. Bertrand Russell states: "A class that has no elements cannot be anything at all." Gottlob Frege also criticized the empty set. Like all intuitionists, Luitzen Brouwer categorically rejects it. Philosophers are also more critical. Edward Jonathan Lowe argues that it is unclear how a set can exist without elements. "The 'empty set' remains an ontological curiosity whose meaning and utility in philosophy is debated."

4. Axiom of "nothing" of sets

In general, "nothing" of things is "no thing". Especially for sets, $\varnothing$ is defined in accordance with reality:
(4) Definition : $\varnothing$ is the sign representing "nothing" of sets, i.e. "no set" and "no element".
The new **axiom of "nothing" of sets** is introduced:
The existence of the sign $\mathcal{S} = \varnothing$ is equivalent to stating for all sets M, that a set M equal to $\varnothing$ does not exist.
(5) Axiom : $\exists \mathcal{S}: \mathcal{S} = \varnothing \leftrightarrow \forall M: \neg (\exists M : M = \varnothing)$

5. Current theory of the transfinite set

For the formal development of the transfinite from the empty set, transfinite induction of John v. Neumann is taken over. It is assumed that $\varnothing$ is the 1st element with cardinal number 0. Considering 0 and 1, it is clear that { 0 } = 1, where 1 is the 2nd element of the natural numbers. Accordingly, it is plausible that $\{ \varnothing \}$ is the second element with ordinal number 1. The union of $\varnothing$ and $\{ \varnothing \}$ then is the 3rd element with ordinal number 2 (6, 7). In general, the successor set is the union of the predecessor set x and the set { x }. M_{trans} then results as the first transfinite set:
The associated ordinal numbers are shown in (7).

(6) $\varnothing$ $\{ \varnothing \}$ $\{ \varnothing, \{ \varnothing \} \}$ $\{\{ \varnothing, \{ \varnothing \}, \{\{ \varnothing, \{ \varnothing \}\}\}$ …......................... Mtrans
(7) 0 1 2 3 .... ω

The **formal representation of the axiom of infinity (8)** thus is understandable:
Set M_{trans} exists: $\varnothing$ is the 1st element of M_{trans} and for all elements x applies: x is an element of M_{trans} implies: x united with { x } is an element of M_{trans}.
(8) $\exists M_{trans}: (\varnothing \in M_{trans} \wedge \forall x: (x \in M_{trans} \rightarrow x \cup \{ x \} \in M_{trans}))$
According to ZFC $\varnothing \in M_{trans}$ is necessary for the transfinite set M_{trans} in the same way as $0 \in \mathbb{N}$ for the transfinite set $\mathbb{N}$ of natural numbers (Article II).

6. Refutation of the transfinite set

That transfinite induction according to (6) is in **contradiction to empiricism,** is obvious if apples are regarded. The set of the empty set would be the second element and the first apple. This is impossible. Existence of the transfinite set is thereby refuted as well.
Formally, non-existence of the "empty set $\varnothing$" according to axiom (5) proves that the axiom of infinity (8) is invalid.
The empty set does not appear in Cantor's theory, so the axiom of infinity contradicts Cantor.

7. Epistemology

Another fundamental objection to the existence of transfinite numbers and sets has already been raised in Article II and is repeated here. **David Hilbert has acknowledged that the transfinite contradicts reality, just being an idea.** Even today, no mathematician or logician will claim that

the transfinite can be demanded without contradiction in accordance with physics and astronomy. Nevertheless, a mathematically consistent theory of the transfinite is postulated.
But epistemology demands that a rationalistic idea that contradicts reality is necessarily false. Immanuel Kant has overcome the contradiction between rationalism and empiricism with his transcendental philosophy. **The transfinite already fails because of Kant's epistemology.**